

Casadio-Fabbri-Mazzacurati Black Strings and Braneworld-induced Quasars Luminosity Corrections

Roldão da Rocha*

*Centro de Matemática, Computação e Cognição,
Universidade Federal do ABC 09210-170, Santo André, SP, Brazil.*

A. Piloyan†

*Institut für Theoretische Physik, Philosophenweg 16 D-6912, Heidelberg, Germany
Yerevan State Univ., Faculty of Physics, Alex Manoogian 1, Yerevan 0025, Armenia*

A. M. Kuerten‡

Centro de Ciências Naturais e Humanas, Universidade Federal do ABC 09210-170, Santo André, SP, Brazil.

C. H. Coimbra-Araújo§

Campus Palotina, Universidade Federal do Paraná, UFPR, 85950-000, Palotina, PR, Brazil.

This paper aims to evince the corrections on the black string warped horizon in the braneworld paradigm, and their drastic physical consequences, as well as to provide subsequent applications in astrophysics. Our analysis concerning black holes on the brane departs from the Schwarzschild case, where the black string is unstable to large-scale perturbation. The cognizable measurability of the black string horizon corrections due to braneworld effects is investigated, as well as their applications in the variation of quasars luminosity. We delve into the case wherein two solutions of Einstein's equations proposed by Casadio, Fabbri, and Mazzacurati, regarding black hole metrics presenting a post-Newtonian parameter measured on the brane. In this scenario, it is possible to analyze purely the braneworld corrected variation in quasars luminosity, by an appropriate choice of the post-Newtonian parameter that precludes Hawking radiation on the brane: the variation in quasars luminosity is uniquely provided by pure braneworld effects, as the Hawking radiation on the brane is suppressed.

PACS numbers: 04.50.Gh, 04.50.-h, 11.25.-w

I. INTRODUCTION

Black holes solutions of Einstein equations in general relativity are useful tools to investigate the space-time structure and underlying models for gravity and its quantum effects, as well as to study the astrophysics regarding supermassive objects, for instance. Extra-dimensional space-times are scenarios for extensions of the general relativity, providing solutions to the Einstein's field equations, as black holes metrics in higher dimensions, and some ensuing applications to cosmology in such a context. In addition, the recent effort to deal with the hierarchy problem, by inducing gravity to leak into extra dimensions [1], is explored in braneworld models. They are based on M-theory and string theory [2–4]. In particular, an useful approach to deal with the hierarchy is provided an effective 5D reduction of the Hořava-Witten theory [3, 5, 6].

Impelled by a thorough development concerning gravity on 5D braneworld scenarios and some applications, in particular the black holes/black strings horizons variations induced by braneworld effects, [7–12], further aspects concerning corrections in the black string like objects and their warped horizons are here introduced. The Casadio-Fabbri-Mazzacurati metrics on the brane, namely the type I and type II black hole solutions [13, 14] are now analyzed and regarded as generating the bulk metric, inducing a black string like warped horizon. This procedure is well known for the Schwarzschild metric [9, 10, 12, 15]. The Casadio-Fabbri-Mazzacurati metrics depart from the Schwarzschild solution, possessing a post-Newtonian parameter. For some particular choice of this parameter, the black hole Hawking radiation on the brane is suppressed [13, 14]. The black holes Hawking radiation in braneworld scenarios was

*Electronic address: roldao.rocha@ufabc.edu.br

†Electronic address: arpine.piloyan@ysu.am

‡Electronic address: andre.kuerten@ufabc.edu.br

§Electronic address: carlos.coimbra@ufpr.br

comprehensively investigated in, e. g., [16, 17].

This article is organized as follows: in Sec. II, after presenting the Einstein field equations in the brane, the deviation in Newton's 4D gravitational potential is revisited. For a static spherical metric on the brane, the propagating effect of 5D gravity is evinced from the Taylor expansion (along the extra dimension) of the metric. Such expansion is accomplished in powers of the normal coordinate — out of the brane — which provides the black string warped horizon profile. Such expansion can provide the bulk metric uniquely from the metric on the brane. In Sec. III, the type I and type II Casadio-Fabbri-Mazzacurati black string solutions and their respective warped horizons are obtained, analyzed and depicted. We analyze such solutions in the particular case where the associated post-Newtonian parameter makes the black hole Hawking radiation to be suppressed. Such analysis has paramount importance, since to measure pure effects to the corrections (by braneworld effects) for quasars luminosity is aimed. In Sec. IV, for an illustrative model for accretion in a supermassive black hole, the variation of luminosity in quasars is investigated more precisely for the two models provided by Casadio-Fabbri-Mazzacurati, and compared to the pure Schwarzschild black string. The correction effects on the black string warped horizon, induced and generated by braneworld models, preclude the Hawking radiation on the brane for the above mentioned suitable choice of the post-Newtonian parameter. All results are illustrated by graphics and figures, and the quasars luminosity provided by the Casadio-Fabbri-Mazzacurati black hole solution is compared with the Schwarzschild one.

II. BLACK STRING BEHAVIOR ALONG THE EXTRA DIMENSION

Hereupon the notation in [15, 18, 19] is adopted, where $\{\theta_\mu\}$, $\mu = 0, 1, 2, 3$, typify a basis for the cotangent space T_x^*M at a point x on a brane M embedded in a bulk. A frame $\theta^A = dx^A$ ($A = 0, 1, 2, 3, 4$) in the bulk is represented in local coordinates. In the brane defined by $y = 0$, [hereon y denotes the associated Gaussian coordinate] $dy = n_A dx^A$ is orthogonal to the brane. The metric $\dot{g}_{AB} dx^A dx^B = g_{\mu\nu}(x^\alpha, y) dx^\mu dx^\nu + dy^2$ endows the bulk, which is related to the brane metric $g_{\mu\nu}$ by $g_{\mu\nu} = \dot{g}_{\mu\nu} - n_\mu n_\nu$. The bulk indexes $A, B = 0, \dots, 3$, as $\dot{g}_{44} = 1$ and $\dot{g}_{\mu 4} = 0$ [15]. Hereupon the standard relations $\Lambda_4 = \frac{\kappa_5^2}{2} \left(\frac{1}{6} \kappa_5^2 \lambda^2 + \Lambda \right)$ and $\kappa_4^2 = \frac{1}{6} \lambda \kappa_5^4$ are considered, where Λ_4 denotes the effective brane cosmological constant, and λ is the brane tension. The constant $\kappa_5 = 8\pi G_5$, where G_5 is the 5D Newton gravitational constant, denotes the 5D gravitational coupling, related to the 4D gravitational constant G by $G_5 = G \ell_{\text{Planck}}$, where $\ell_{\text{Planck}} = \sqrt{G \hbar / c^3}$ is the Planck length. The junction condition provides the extrinsic curvature tensor $K_{\mu\nu} = \frac{1}{2} \mathcal{L}_n g_{\mu\nu}$ by [15, 20]

$$K_{\mu\nu} = -\frac{1}{2} \kappa_5^2 \left(T_{\mu\nu} + \frac{1}{3} (\lambda - T) g_{\mu\nu} \right), \quad (1)$$

where $T = T^\mu_\mu$ is the trace of the energy-momentum tensor. The 5D Weyl tensor is given by $C_{\mu\nu\sigma\rho} = {}^{(5)}R_{\mu\nu\sigma\rho} - \frac{2}{3} (\dot{g}_{[\mu\sigma} {}^{(5)}R_{\nu]\rho} + \dot{g}_{[\nu\rho} {}^{(5)}R_{\mu]\sigma}) - \frac{1}{6} {}^{(5)}R (\dot{g}_{\mu[\sigma} \dot{g}_{\nu]\rho})$, where ${}^{(5)}R_{\mu\nu\sigma\rho}$ denotes the components of the bulk Riemann tensor (as usual ${}^{(5)}R_{\mu\nu}$ and ${}^{(5)}R$ are the associated Ricci tensor and the scalar curvature). The trace-free and symmetric components, respectively denoted by $\mathcal{B}_{\mu\nu\alpha} = g_\mu^\rho g_\nu^\sigma C_{\rho\sigma\alpha\beta} n^\beta$ and $\mathcal{E}_{\mu\nu} = C_{\mu\nu\sigma\rho} n^\sigma n^\rho$, denote the magnetic and electric Weyl tensor components, respectively.

A. Brane field equations

The Einstein brane field equations can be expressed as

$$G_{\mu\nu} = -\mathcal{E}_{\mu\nu} - \frac{1}{2} \Lambda_5 g_{\mu\nu} + \frac{1}{4} \kappa_5^4 \left(\frac{1}{2} g_{\mu\nu} (T^2 - T_{\alpha\beta} T^{\alpha\beta}) + T T_{\mu\nu} - T_{\mu\alpha} T_\nu^\alpha \right).$$

The Weyl tensor electric term $\mathcal{E}_{\mu\nu}$ carries an imprint of high-energy effects sourcing Kaluza-Klein (KK) modes. The gravitational potential $V(r) = \frac{GM}{c^2 r}$, associated to the 4D classical gravity, is corrected by extra-dimensional effects [5, 15]:

$$V(r) = \frac{GM}{c^2 r} \left(1 + \frac{2\ell^2}{3r^2} + \dots \right). \quad (2)$$

The parameter ℓ is associated with the bulk curvature radius and corresponds to the effective size of the extra dimension probed by a 5D graviton [5, 15, 21]. Indeed, the contribution of the massive KK modes sums to a correction of the

4D potential. At small scales $r \ll \ell$, one obtains the 5D features related to the potential $V(r) \approx GM\ell/r^2$. For $r \gg \ell$ the potential is provided by (2) reinforcing the gravitational field [5, 10, 15].

Considering vacuum on the brane, where $T_{\mu\nu} = 0$ outside a black hole, the field equations $G_{\mu\nu} = -\mathcal{E}_{\mu\nu} - \frac{1}{2}\Lambda_5 g_{\mu\nu}$ and $R = 0 = \mathcal{E}_{\mu}^{\mu}$ hold for braneworlds with \mathbb{Z}_2 -symmetry. The vacuum field equations in the brane are $\mathcal{E}_{\mu\nu} = -R_{\mu\nu}$, where the bulk cosmological constant is comprised into the warp factor. The bulk can host for instance a plethora of non standard model fields, as dilatonic or moduli fields [22]. Although a Taylor expansion of the metric was used to probe properties of a black hole on the brane in, e. g., [15, 23], in order to enhance the range of our analysis throughout this paper a more complete approach to analyze braneworld corrections in the black string profile can be accomplished, based on [12].

A Taylor expansion of the metric along the extra dimension allows us to analyze the black string more deeply. The effective field equations are complemented by other ones, obtained from the 5D Einstein and Bianchi equations in Refs. [15, 18, 19]. Hereupon, since we are concerned with the Taylor expansion of the metric along the extra dimension up to the fourth order, besides the effective field equation $\mathcal{L}_{\mathbf{n}} K_{\mu\nu} = K_{\mu\alpha} K^{\alpha}_{\nu} - \mathcal{E}_{\mu\nu} - \frac{1}{6}\Lambda_5 g_{\mu\nu}$, the effective equations are considered:

$$\begin{aligned}\mathcal{L}_{\mathbf{n}} \mathcal{E}_{\mu\nu} &= \nabla^{\alpha} \mathcal{B}_{\alpha(\mu\nu)} + (K_{\mu\alpha} K_{\nu\beta} - K_{\alpha\beta} K_{\mu\nu}) K^{\alpha\beta} + K^{\alpha\beta} R_{\mu\alpha\nu\beta} - K \mathcal{E}_{\mu\nu} + \frac{1}{6}\Lambda_5 (K_{\mu\nu} - g_{\mu\nu} K) \\ &\quad + 3K^{\alpha}_{(\mu} \mathcal{E}_{\nu)\alpha}, \\ \mathcal{L}_{\mathbf{n}} \mathcal{B}_{\mu\nu\alpha} &= K_{\alpha}^{\beta} \mathcal{B}_{\mu\nu\beta} - 2\nabla_{[\mu} \mathcal{E}_{\nu]\alpha} - 2\mathcal{B}_{\alpha\beta[\mu} K_{\nu]}^{\beta}.\end{aligned}$$

These expressions are used to compute the terms in the Taylor expansion of the metric, along the extra dimension, providing the black string profile and further physical consequences as well. The effective field equations above were employed to construct a covariant analysis of the weak field [18]. Denoting $K = K^{\mu}_{\mu}$, the Taylor expansion is given by [12] [hereon we denote $g_{\mu\nu}(x, 0) = g_{\mu\nu}$]:

$$\begin{aligned}g_{\mu\nu}(x, y) &= g_{\mu\nu}(x, 0) - \kappa_5^2 \left[T_{\mu\nu} + \frac{1}{3}(\lambda - T)g_{\mu\nu} \right] |y| + \\ &\quad + \left[-\mathcal{E}_{\mu\nu} + \frac{1}{4}\kappa_5^4 \left(T_{\mu\alpha} T^{\alpha}_{\nu} + \frac{2}{3}(\lambda - T)T_{\mu\nu} \right) + \frac{1}{6} \left(\frac{1}{6}\kappa_5^4(\lambda - T)^2 - \Lambda_5 \right) g_{\mu\nu} \right] y^2 + \\ &\quad + \left[2K_{\mu\beta} K^{\beta}_{\alpha} K^{\alpha}_{\nu} - (\mathcal{E}_{\mu\alpha} K^{\alpha}_{\nu} + K_{\mu\alpha} \mathcal{E}^{\alpha}_{\nu}) - \frac{1}{3}\Lambda_5 K_{\mu\nu} - \nabla^{\alpha} \mathcal{B}_{\alpha(\mu\nu)} + \frac{1}{6}\Lambda_5 (K_{\mu\nu} - g_{\mu\nu} K) \right. \\ &\quad \left. + K^{\alpha\beta} R_{\mu\alpha\nu\beta} + 3K^{\alpha}_{(\mu} \mathcal{E}_{\nu)\alpha} - K \mathcal{E}_{\mu\nu} + (K_{\mu\alpha} K_{\nu\beta} - K_{\alpha\beta} K_{\mu\nu}) K^{\alpha\beta} - \frac{\Lambda_5}{3} K_{\mu\nu} \right] \frac{|y|^3}{3!} + \\ &\quad + \left[\frac{\Lambda_5}{6} \left(R - \frac{\Lambda_5}{3} + K^2 \right) g_{\mu\nu} + \left(\frac{K^2}{3} - \Lambda_5 \right) K_{\mu\alpha} K^{\alpha}_{\nu} + (3K^{\alpha}_{\mu} K^{\beta}_{\alpha} - K_{\mu\alpha} K^{\alpha\beta}) \mathcal{E}_{\nu\beta} \right. \\ &\quad + (K^{\alpha}_{\sigma} K^{\sigma\beta} + \mathcal{E}^{\alpha\beta} + K K^{\alpha\beta}) R_{\mu\alpha\nu\beta} - \frac{1}{6}\Lambda_5 R_{\mu\nu} + 2K_{\mu\beta} K^{\beta}_{\sigma} K^{\sigma}_{\alpha} K^{\alpha}_{\nu} + K_{\sigma\rho} K^{\sigma\rho} K K_{\mu\nu} \\ &\quad + \mathcal{E}_{\mu\alpha} \left(K_{\nu\beta} K^{\alpha\beta} - 3K^{\alpha}_{\sigma} K^{\sigma}_{\nu} + \frac{1}{2}K K^{\alpha}_{\nu} \right) + \left(\frac{7}{2}K K^{\alpha}_{\mu} - 3K^{\alpha}_{\sigma} K^{\sigma}_{\mu} \right) \mathcal{E}_{\nu\alpha} - \frac{13}{2}K_{\mu\beta} \mathcal{E}^{\beta}_{\alpha} K^{\alpha}_{\nu} \\ &\quad \left. + (R - \Lambda_5 + 2K^2) \mathcal{E}_{\mu\nu} - K_{\mu\alpha} K_{\nu\beta} \mathcal{E}^{\alpha\beta} - \frac{7}{6}K^{\sigma\beta} K^{\alpha}_{\mu} R_{\nu\sigma\alpha\beta} - 4K^{\alpha\beta} R_{\mu\nu\gamma\alpha} K^{\gamma}_{\beta} \right] \frac{|y|^4}{4!} + \dots\end{aligned}\tag{3}$$

Such an expansion was analyzed in [15, 24] only up to the second order, although it fizzled out to explain more reliably the black string horizon behavior along the extra dimension. In addition, this higher order expansion provides further physical features regarding variable tension braneworld scenarios, since the expansion terms beyond second order provide drastic modifications in the stability of black strings [12]. For an alternative method which does not take into account the \mathbb{Z}_2 -symmetry, and some subsequent applications, see [25].

For a vacuum in the brane, Eq.(3) reads

$$\begin{aligned}
g_{\mu\nu}(x, y) = & g_{\mu\nu} - \frac{1}{3}\kappa_5^2\lambda g_{\mu\nu} |y| + \left[\frac{1}{6} \left(\frac{1}{6}\kappa_5^4\lambda^2 - \Lambda_5 \right) g_{\mu\nu} - \mathcal{E}_{\mu\nu} \right] y^2 \\
& - \frac{1}{6} \left(\left(\frac{193}{36}\lambda^3\kappa_5^6 + \frac{5}{3}\Lambda_5\kappa_5^2\lambda \right) g_{\mu\nu} + \kappa_5^2 R_{\mu\nu} \right) \frac{|y|^3}{3!} + \\
& + \left[\frac{1}{6}\Lambda_5 \left(\left(R - \frac{1}{3}\Lambda_5 - \frac{1}{18}\lambda^2\kappa_5^4 \right) + \frac{7}{324}\lambda^4\kappa_5^8 \right) g_{\mu\nu} + \left(R - \Lambda_5 + \frac{19}{36}\lambda^2\kappa_5^4 \right) \mathcal{E}_{\mu\nu} \right. \\
& \left. + \frac{1}{6} \left(\frac{37}{36}\lambda^2\kappa_5^4 - \Lambda_5 \right) R_{\mu\nu} + \mathcal{E}^{\alpha\beta} R_{\mu\alpha\nu\beta} \right] \frac{y^4}{4!} + \dots
\end{aligned} \tag{4}$$

This expression is shown to be prominently relevant for our subsequent analysis.

Hereon, the black hole horizon evolution along the extra dimension — the warped horizon [26] — shall be investigated, exploring the component $g_{\theta\theta}(x, y)$ in (4). Indeed, let us consider any spherically symmetric metric associated to a black hole — in particular the Schwarzschild and the Casadio-Fabbri-Mazzacurati ones here investigated. Such metric has the radial coordinate given by $\sqrt{g_{\theta\theta}(x, 0)} = r$. The black hole solution, namely, the black string solution *on the brane*, is regarded when $\sqrt{g_{\theta\theta}(x, 0)} = R$, where R denotes the coordinate singularity, usually calculated by the component $g_{rr}^{-1} = 0$ in the metric¹. In the Schwarzschild metric $R = R_S = \frac{2GM}{c^2r}$. The coordinate singularities for the Casadio-Fabbri-Mazzacurati metrics are going to be analyzed in what follows, in the black string context as well. Such singularities shall be shown to be also physical singularities (associated to the black holes and the black strings as well), by analyzing their respective four- and 5D Kretschmann scalars. In other words, in the analysis regarding the black string behavior along the extra dimension, we are concerned merely about the warped horizon behavior, which is provided uniquely by the value for the metric on the brane $\sqrt{g_{\theta\theta}(x, y)}|_{r=R_S}$. More specifically, the black string horizon for the Schwarzschild metric — or warped horizon [26] — is defined when the radial coordinate r has the value $r = R_S = \frac{2GM}{c^2}$, which is obtained when the coefficient $(1 - \frac{2GM}{c^2r}) = g_{rr}$ of the term dr^2 in the metric goes to infinity [16]. It corresponds to the black hole horizon on the brane. On the another hand, the (squared) general radial coordinate in spherical coordinates legitimately appears as the term $g_{\theta\theta}d\theta^2 = r^2d\theta^2$ in the Schwarzschild metric. Our analysis of the term $g_{\theta\theta}(x, y)$ (given by Eq.(3) for $\mu = \theta = \nu$ as the most general case, and provided by Eq.(4) for the Schwarzschild metric) holds for any value r . In particular, the term originally coined “black string” corresponds to the Schwarzschild case [26], defined by the black hole horizon evolution along the extra dimension into the bulk. Hence, the black string regards solely the so called “warped horizon”, which is $g_{\theta\theta}(x, y)$, for the particular case where $r = R_S$ is a coordinate singularity.

III. CASADIO-FABBRI-MAZZACURATI BRANEWORLD SOLUTIONS

The analysis of the gravitational field equations on the brane is not straightforward, due to the fact that the propagation of gravity into the bulk does not allow a complete presentation of the brane gravitational field equations as a closed form system [18]. The investigation concerning the gravitational collapse on the brane is therefore very complicated [27]. The solutions provided by Casadio, Fabbri, and Mazzacurati for the brane black holes metrics [13, 23, 28] take into account the post-Newtonian parameter β , measured on the brane. The case $\beta = 1$ generates forthwith an exact Schwarzschild solution on the brane, and elicits a black string prototype. Furthermore, it was observed in [13, 14] that $\beta \approx 1$ holds in solar system scales measurements [24]. The parameter β is, furthermore, capable to indicate and to measure the difference between the inertial mass and the gravitational mass of a test body. This parameter also affects the perihelion shift and provides the Nordtvedt effect [24]. Moreover, measuring β gives information about the vacuum energy of the braneworld or, equivalently, the cosmological constant [13, 14, 29].

One of the main motivation regarding the Casadio-Fabbri-Mazzacurati setup is that black holes solutions of the Einstein equations on the brane must depart from the Schwarzschild solution. In particular, the Schwarzschild associated black string is unstable to large-scale perturbations [30, 31]: the associated Kretschmann scalar, regarding the 5D curvature, diverges on the Cauchy horizon [13, 15]. Indeed, it is important to emphasize that, as we shall see for the Casadio-Fabbri-Mazzacurati black string, it might be possible to find out points along the extra dimension for which the Kretschmann scalar ${}^{(5)}K = {}^{(5)}R_{\mu\nu\rho\sigma}{}^{(5)}R^{\mu\nu\rho\sigma}$ diverges, i. e., they are indeed naked singularities along

¹ Such calculation was also considered in Eq.(II.6) of [16] in the braneworld context.

the extra dimension. For instance, in order to identify singularities, for a Schwarzschild black string $(^5)K \propto 1/r^6$ [31]. Hence there is a line singularity at $r = 0$ along the extra dimension, but not at the Schwarzschild horizon [15, 26]. Since the pure black string configuration is unstable [30], this structure is not physical ab initio. Anyway for $y = 0$ one reproduces the Kretschmann scalars standard 4D behavior. For the Schwarzschild solution, the singularity on the brane extends into the bulk and makes the AdS horizon singular. The Casadio-Fabbri-Mazzacurati black string solutions and their respective braneworld corrections are going to be presented and their stability analyzed as well. For the sake of completeness the next Section is briefly devoted to the braneworld corrections to the Schwarzschild solution [7, 15].

A static spherical metric on the brane is provided by $g_{\mu\nu}dx^\mu dx^\nu = -F(r)dt^2 + (H(r))^{-1}dr^2 + r^2d\Omega^2$. The Schwarzschild metric is corresponds to $F(r) = H(r) = 1 - \frac{2GM}{c^2r}$. Obtaining such functions remains an open problem in the black hole theory on the brane [13–15]. Considering the Weyl tensor projected electric component on the brane $\mathcal{E}_{\theta\theta} = 0$ [15] yields [12]

$$g_{\theta\theta}(r, y) = r^2 \left[1 - \frac{\kappa_5^2 \lambda}{3} |y| + \frac{1}{6} \left(\frac{1}{6} \kappa_5^4 \lambda^2 - \Lambda_5 \right) y^2 - \left(\frac{193}{216} \lambda^3 \kappa_5^6 + \frac{5}{18} \Lambda_5 \kappa_5^2 \lambda \right) \frac{|y|^3}{3!} + \right. \\ \left. - \frac{1}{18} \Lambda_5 \left(\left(\Lambda_5 + \frac{1}{6} \lambda^2 \kappa_5^4 \right) + \frac{7}{324} \lambda^4 \kappa_5^8 \right) \frac{y^4}{4!} + \dots \right] \quad (5)$$

Note that obviously in the brane $g_{\theta\theta}(r, 0) = r^2$. Defining $\psi(r)$ as the deviation from a Schwarzschild form for $H(r)$ [14, 15, 32–36] as $H(r) = 1 - \frac{2GM}{c^2r} + \psi(r)$, for a large black hole with horizon scale $r \gg \ell$ it follows from Eq.(2) that

$$\psi(r) \approx -\frac{4GM\ell^2}{3c^2r^3}. \quad (6)$$

The formula above, together with Eq.(2), can be forthwith derived from the RS analysis concerning small gravitational fluctuations in terms of KK modes, where a curved background can support a bound state of the higher-dimensional graviton [5, 37]. Besides, the effect of the KK modes on the metric outside a specific matter distribution on the brane was incorporated by [37] in the form of the $1/r^3$ correction to the gravitational potential. Such corrections in the inverse-square law were experimentally shown in [38].

Now, given a general static spherically symmetric

$$g_{\mu\nu}dx^\mu dx^\nu = -N(r)dt^2 + A(r)dr^2 + r^2d\Omega^2, \quad (7)$$

the Casadio-Fabbri-Mazzacurati 4D black hole solution was obtained in [13, 14]. The Schwarzschild 4D metric is obtained when $N(r) = (A(r))^{-1}$ and $N(r) = 1 - \frac{2GM}{c^2r}$. Its unique extension into the bulk is a black string warped horizon, with the central singularity extending all along the extra dimension, and the bulk horizon singular [13, 14, 31]. If the Schwarzschild metric on the brane is demanded with a regular AdS horizon, there is no matter confinement on the brane: in this case matter percolates into the bulk [36]. The condition $N(r) = (A(r))^{-1}$ holds in the 4D case, although the most general solution is the Reissner-Nordström one [13, 14, 18], related to the case II analyzed in which follows. The case I below concerns the function $N(r) = 1 - \frac{2GM}{c^2r}$ like the Schwarzschild case, but this time $A(r)$ to be calculated. Both cases are profoundly investigated, as well as their prominent applications.

A. Casadio-Fabbri-Mazzacurati black string: Case I

This case was analyzed by Casadio, Fabbri, and Mazzacurati in [13, 14], regarding the 4D black hole solution (7). They obtained a solution of the Einstein's equations distinguished from the Schwarzschild one, provided by the metric coefficients

$$N(r) = 1 - \frac{2GM}{c^2r} \quad \text{and} \quad A(r) = \frac{1 - \frac{3GM}{2c^2r}}{\left(1 - \frac{2GM}{c^2r}\right) \left(1 - \frac{GM}{2c^2r}(4\beta - 1)\right)} \quad (8)$$

to be considered in (7). The solution (8) depends on just one parameter and for $M \rightarrow 0$ one recovers the Minkowski vacuum.

The Casadio-Fabbri-Mazzacurati black string classical horizon, in the brane, is the solution of the algebraic equation $(A(r))^{-1} = 0$. In order to extract phenomenological information of numerical calculations, we first consider in this Subsection the case where $\beta = 5/4$. Indeed, our aim is to analyze the pure braneworld corrected effects on the variation of luminosity in quasars, composed by a black hole which presents Hawking radiation in the brane equal to

zero [13, 14]. It makes feasible our analysis on the variation of quasars luminosity, purely due to braneworld effects. Hawking radiation in the context of black strings was investigated, e.g., in [16, 39]. Note that the metric above was also derived as a possible geometry outside a star on the brane [28]. The corresponding Hawking temperature is calculated in [13]. In comparison with Schwarzschild black holes, the black hole provided by this solution is either hotter or colder, depending upon the sign of $(\beta - 1)$.

The extension of these solutions into the bulk has prominent importance addressed in [13]. For the Schwarzschild case, the singularity on the brane extends into the bulk and makes the AdS horizon singular. Notwithstanding, according to the analysis illustrated by the graphics below, Eq.(3) asserts that the black string solutions might be regular for supermassive black holes.

Taking into account the metric in (7), the classical standard black hole radius is given by — supposing $r \neq \frac{3GM}{2c^2}$ — two solutions of Schwarzschild type $R_S = \frac{2GM}{c^2}$, for our choice of the parameter $\beta = 5/4$, providing zero Hawking black hole temperature. The assumption $r \neq 3GM/c^2$ is quite natural: for this case the 4D Kretschmann scalar $K^{(I)} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverges for $r = 0$ and $r = \frac{3GM}{2c^2}$ (see the Appendix). Now, the Gauss equation is well known to relate the 5D and the 4D Riemann curvature tensor as

$${}^{(5)}R_{\nu\rho\sigma}^{\mu} = R_{\nu\rho\sigma}^{\mu} - K_{\rho}^{\mu}K_{\nu\sigma} + K_{\sigma}^{\mu}K_{\nu\rho}. \quad (9)$$

By taking the junction conditions into account, where consequently $K_{\mu\nu} = -\frac{1}{2}\kappa_5^2(T_{\mu\nu} + \frac{1}{3}(\lambda - T)g_{\mu\nu})$, for the vacuum case here considered it follows that $K_{\mu\nu} = -\frac{1}{6}\kappa_5^2\lambda g_{\mu\nu}$. By inserting it in the Gauss equation (9), it implies that the 5D Kretschmann scalar ${}^{(5)}K = {}^{(5)}R_{\mu\nu\rho\sigma}{}^{(5)}R^{\mu\nu\rho\sigma}$ for the Casadio-Fabbri-Mazzacurati type I black string also diverges for $r = 0$ and $r = \frac{3GM}{2c^2}$: the terms involving the extrinsic curvature in (9) above are not capable to cancel the divergence provided by the 4D Kretschmann scalar, in the computation for ${}^{(5)}K$.

Using the same procedure as [10, 15], one can use the metric coefficients (8) in Eq.(4) and calculate the black string warped horizon. As asserted, for instance in [13, 14], this analysis can be attempted either numerically or by Taylor expanding all 5D metric elements in powers of the extra coordinate. In the graphics below, we explicit the value for the black string warped horizon, provided by $\sqrt{g_{\theta\theta}(R_S, y)}$, where R_S is the Schwarzschild radius. Further, $\lambda = \Lambda = 1 = \kappa_5$ hereupon [M_\odot denotes the sun mass]:

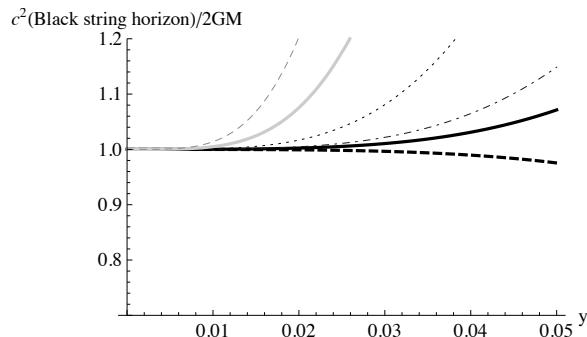


FIG. 1: Graphic of the brane effect-corrected black string horizon $\sqrt{g_{\theta\theta}(R_S, y)}$ in the Casadio-Fabbri-Mazzacurati first solution, along the extra dimension y , for different values of the black hole mass M . For the dash-dotted line $M = M_\odot$; for the black dashed line: $M = 10 M_\odot$; for the thick black line: $M = 10^2 M_\odot$; for the black dotted line: $M = 10^3 M_\odot$; for the thick gray line $M = 10^4 M_\odot$; for the gray dotted line $M = 10^5 M_\odot$.

Fig. 1 evinces a very interesting profile for the black string horizon behavior along the extra dimension y in gaussian coordinates. It indicates a critical mass M (indeed our simulations provide $M \sim 73 M_\odot$) above which the associated black string warped horizon monotonically increases along the extra dimension. The black string is known to be placed in the bulk, in a tubular neighborhood along the axis of symmetry. A singularity associated to the black string is a fixed point y_0 (fixed) in the axis of symmetry along the extra dimension, such that the black string transversal slice has radius equal to zero. We show here that at the coordinate singularities $r = 0$ and $r = \frac{3GM}{2c^2}$ there is a physical singularity for the black string at such values, irrespective of the value for y . In fact, the Kretschmann scalar $K = {}^{(5)}R_{\mu\nu\rho\sigma}{}^{(5)}R^{\mu\nu\rho\sigma}$ diverges for such values (see the Appendix). Notwithstanding, the black string warped horizon $\sqrt{g_{\theta\theta}(R_S, y)}$ does not equal to zero, as illustrated at the Fig. 1.

B. Casadio-Fabbri-Mazzacurati black string: Case II

An alternative solution of (7) is obtained in [13, 14] where the metric coefficients

$$N(r) = 1 - \frac{2GM}{c^2r} + \frac{2G^2M^2}{c^4r^2}(\beta - 1), \quad A(r) = \frac{1 - 3GM/2c^2r}{(1 - \frac{2GM}{c^2r})(1 - \frac{GM}{2c^2r}(4\beta - 1))} \quad (10)$$

are considered in (7). In order that the Hawking temperature be zero on the brane, the choice $\beta = 3/2$ is demanded [13]. The classical solution R for the black hole horizon is given by $R = R_S$ and $R = 5R_S/2$, where R_S denotes the Schwarzschild radius. It implies that the black string horizon now corrected by braneworld effects when (10) is substituted in (4), providing the graphic below.

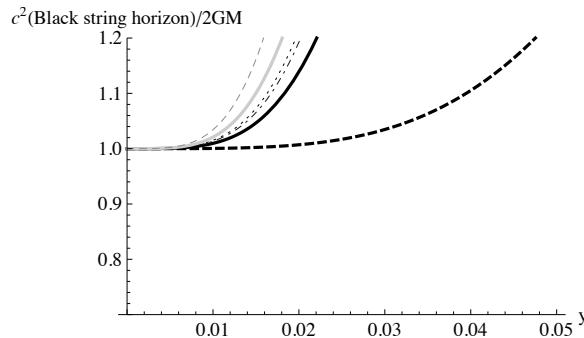


FIG. 2: Graphic of the brane effect-corrected Casadio-Fabbri-Mazzacurati type II black string horizon $\sqrt{g_{\theta\theta}(R_S, y)}$, along the extra dimension y , for different values of the black hole mass GM/c^2 in the brane. For the black dashed line $M = M_\odot$; for the gray line $M = 10 M_\odot$; for the black line: $M = 10^2 M_\odot$; the dash-dotted line: $M = 10^3 M_\odot$; for the dotted line: $M = 10^4 M_\odot$; for the gray dashed line $M = 10^5 M_\odot$; for the thick gray line $M = 10^6 M_\odot$.

The black string horizon profile along the extra dimension is qualitatively similar for all values of M depicted here: the warped horizon always increases monotonically. Furthermore, under a similar analysis accomplished this time for the case II Casadio-Fabbri-Mazzacurati, and by taking into account the Kretschmann scalar (A2) in the Appendix, we conclude that such expression diverges for $r = \frac{3GM}{2c^2}$, for $r = \frac{5GM}{2c^2}$ and $r = \frac{2GM}{c^2}$. Contrary to the Schwarzschild metric, which presents the black hole horizon as a coordinate singularity — which can circumvented by, e. g., the Kruskal-Szekeres coordinates — and not as a physical singularity, the Kretschmann scalar for the Casadio-Fabbri-Mazzacurati case II metric indicates that each black hole horizon on the brane is a physical singularity, since it diverges for such values. Again, the terms involving the extrinsic curvature in (9) above are not able to cancel the divergence induced by the 4D Kretschmann scalar, when one calculates ${}^{(5)}K$. Hence, the black string also diverges for such values.

Fig. 2 indicates that the Casadio-Fabbri-Mazzacurati (case II) black string horizon always increases. Since the bulk has no fixed metric a priori, but it can be calculated from (3) taking into account the metric on the brane, we can calculate the bulk curvature using the metric coefficients in (3).

Compact sources on the brane, such as stars and black holes, have been investigated extensively. However, their description has proven rather complicated and there is little hope to obtain analytic solutions. The present literature does in fact provide solutions on the brane [13, 23, 24, 28], perturbative results over the Randall-Sundrum background [37, 42], and numerical treatments [29]. In [43] the luminosity dissipation, the conditions for which a collapsing star generically evaporates and approaches the Hawking behavior as the (apparent) horizon is formed, are also analyzed.

IV. CORRECTIONS IN THE LUMINOSITY: BRANEWORLD EFFECTS

Once the black string behavior was previously analyzed along the extra dimension, we hereon aim to focus on the corrections now restricted to the phenomena on the brane. These corrections are shown here to induce dramatic consequences on the quasars luminosity variation, due to the braneworld model considered. Due to its prominent importance on the analysis hereupon, the effect of higher dimensions in the gravity sector might begin to make their presence felt as the black hole horizon is approached. The case of braneworld black holes horizon corrections is explored hereon.

Quasars are astrophysical objects that can be found at large astronomical distances. Supermassive stars and the process of gravitational collapse are showed to be able to probe deviations from the 4D general relativity [10]. The

observation of quasars in X -ray band can constrain the measure of the bulk curvature radius ℓ . Varied values for ℓ were used and tested, and no qualitative deviations have been detected. Table-top tests of Newton's law currently find no deviations down to the order $\ell \lesssim 0.1$ mm. A more accurate magnitude limit improvement on the AdS_5 curvature ℓ is provided in [15, 40] by analyzing the existence of stellar-mass black holes on long time scales and of black hole X -ray binaries. Furthermore, the failure of current experiments using torsion pendulums and mechanical oscillators to observe departures from Newtonian gravity at small scales have set the upper limit of ℓ in the region $\ell \lesssim 0.2$ mm [41].

Regarding a static black hole being accreted, in a straightforward model the accretion efficiency η is given by

$$\eta = \frac{GM}{6c^2 R_S}, \quad (11)$$

where R_S denotes the black hole horizon, namely the black string horizon in the brane). The event horizon of the supermassive black hole is 10^{15} times bigger than the bulk curvature parameter ℓ . This is not the case of mini black holes wherein the event horizon of magnitude orders smaller than ℓ . As proved in [10], the solution above for the black string horizon can be also found in terms of the curvature radius ℓ [9]. In the accretion rate model in [44], observational data for the luminosity L estimates a value for ℓ . The luminosity L , due to the accretion in a black hole composing a quasar, is a function of the bulk curvature radius parameter ℓ , and provided by

$$L(\ell) = \eta(\ell) \dot{M} c^2, \quad (12)$$

where \dot{M} denotes the mass accretion rate. For a typical black hole of $10^{12} M_\odot$ in a supermassive quasar, the accretion rate is $\dot{M} \approx 2.1 \times 10^{19} \text{ kg s}^{-1}$ [10]. Supposing that the quasar radiates in the Eddington limit [44] $L = L_{\text{Edd}} \sim 1.2 \times 10^{45} \left(\frac{M}{10^7 M_\odot} \right) \text{ erg s}^{-1}$, the luminosity is given by $L \sim 10^{47} \text{ erg s}^{-1}$. From (11) and (12), the variation in the luminosity of a quasar composed by a supermassive black hole reads

$$\Delta L = \frac{GM}{6c^2} (R_{\text{brane}}^{-1} - R_S^{-1}) \dot{M} c^2 = \frac{1}{12} \left(\frac{R_S}{R_{\text{brane}}} - 1 \right) \dot{M} c^2, \quad (13)$$

where $R_{\text{brane}} = \sqrt{g_{\theta\theta}(r = R_S, y)}$ denotes the black string corrected horizon. In the next Subsection the variation of the quasars luminosity for the two Casadio-Fabbri-Mazzacurati black holes are depicted and analyzed. Furthermore, the difference in the luminosity between the pure Schwarzschild and the case of the solutions (8) and (10) are computed and discussed in what follows.

A. Corrections in the quasar luminosity for the both Casadio-Fabbri-Mazzacurati solutions

We want now to analyze how the corrections for the metric coefficients due to braneworld effects in [5, 9, 10, 15] can affect the luminosity emitted by quasars composed by black holes provided by the Casadio-Fabbri-Mazzacurati solutions (7), with coefficients (8, 10). The alteration in the black hole horizon definitely modifies the quasar luminosity. Its variation with respect to the pure Schwarzschild luminosity is provided by Eq.(13) and here depicted, for the Casadio-Fabbri-Mazzacurati type I metric:

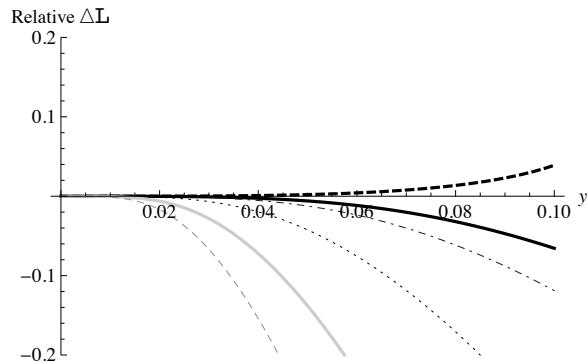


FIG. 3: Graphic of the relative variation of the luminosity $\Delta L/\dot{M}c^2$ in Casadio-Fabbri-Mazzacurati type I model as function of the black hole mass on the brane. For the dashed black line: $M = 10^7 M_\odot$; for the continuous black line: $M = 10^6 M_\odot$; for the dash-dotted line: $M = 10^5 M_\odot$; for the dark dotted line: $M = 10^4 M_\odot$; for the light-gray thick line: $M = 10^3 M_\odot$; for the gray dashed line $M = 10^2 M_\odot$.

Now the Casadio-Fabbri-Mazzacurati case II in Subsec. III B is analyzed, still adopting $\beta = 3/2$ in order to prevent Hawking radiation on the brane. It follows that similarly for the case I analyzed above, the corrected black string horizon on the brane (namely the black hole horizon). One can show that for the corrections on the black string horizon, as seen from the brane, $\Delta L \sim 10^{30 \pm 1} \text{ erg s}^{-1}$, for a typical supermassive black hole with $M \approx 10^9 M_\odot$ (case I and II respectively in the preceding Subsections III A and III B). The figures below regard respectively the cases I and II of Casadio-Fabbri-Mazzacurati black strings horizon on the brane, analyzed in Sec. III. In solar luminosity units $L_\odot \sim 3.9 \times 10^{33} \text{ erg s}^{-1}$, the variation of luminosity of a supermassive black hole quasar due to the correction of the horizon in a braneworld scenario is given by $\Delta L \sim 10^{-3} L_\odot$. Naturally, this small but cognizable correction in the horizon of supermassive black holes implies a consequent correction in the quasar luminosity, via accretion mechanism. This correction is clearly regarded in the luminosity integrated in all wavelength. The detection of these corrections works in particular selected wavelengths, since quasars emit radiation in the soft/hard X-ray band. We remark that the Schwarzschild braneworld corrected black string horizon on the brane was previously investigated in [10], and in that case $\Delta L \sim 10^{-5} L_\odot$.

In addition, the variation of the quasar luminosity regarding the Casadio-Fabbri-Mazzacurati type II model, with respect to the pure Schwarzschild luminosity, is provided by Eq.(13) and here illustrated:

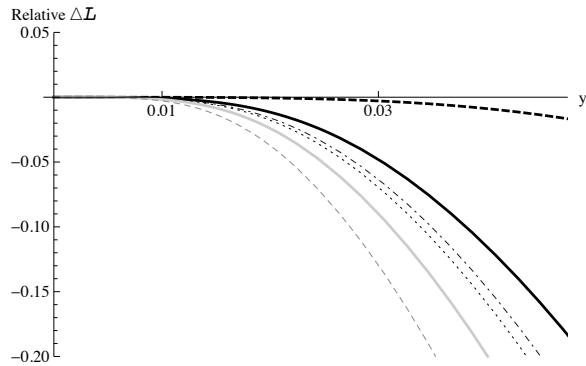


FIG. 4: Graphic of the relative variation of the luminosity $\Delta L/Mc^2$ in Casadio-Fabbri-Mazzacurati type II model as function of the black hole mass on the brane. For the continuous black thin line: $M = 10^6 M_\odot$; for the dash-dotted line: $M = 10^5 M_\odot$; for the light-gray thick line: $M = 10^4 M_\odot$; for the gray dashed line $M = 10^3 M_\odot$; for the dark-gray thick line: $M = 10^2 M_\odot$.

Figs. 3 and 4 evince the variation in the quasars luminosity in the Casadio-Fabbri-Mazzacurati types I and II metrics, respectively Eqs. (8, 10), with respect to the Schwarzschild black hole luminosity. The graphics reveal that the luminosity variation of the Casadio-Fabbri-Mazzacurati black holes, corrected by braneworld effects, is smaller compared to the Schwarzschild case. The exception is the curve in Figure 3 above the horizontal axis, which illustrates the general behavior of the Casadio-Fabbri-Mazzacurati type I black string warped horizon, associated to a black hole mass $M \lesssim 73 M_\odot$.

Delving into the analysis concerning the figures above, the general different profile between the Casadio-Fabbri-Mazzacurati black string warped horizon and the Schwarzschild horizon is expected. Their ratio(s) provides the Figures 3 and 4 by Eq.(13) and the profile is encrypted in the underlying structure of Eq. (4). Indeed, the warped horizon is provided by $\sqrt{g_{\theta\theta}(R_S, y)}$ in (4) when $\mu = \nu = \theta$. Besides, for the Schwarzschild metric the electric component of the Weyl tensor $\mathcal{E}_{\theta\theta}$ equals zero, what do not happen to the Casadio-Fabbri-Mazzacurati metrics. Indeed, taking into account the metric (7), in general $\mathcal{E}_{\theta\theta} = -1 + \frac{1}{A} + \frac{r}{2A} \left(\frac{F'}{F} - \frac{A'}{A} \right) \neq 0$ for the coefficients in (8) and (10).

As it is comprehensively discussed in [45–48], the black hole may recoil away from the brane by the emission of Hawking radiation into the bulk, but not on the brane. We would like to emphasize that *only* mini black holes in the Randall-Sundrum model are prevented to recoil away from the brane into the bulk [46]. Notwithstanding, here the Randall-Sundrum model is not required and all information about the bulk can be extracted from the Casadio-Fabbri-Mazzacurati metrics on the brane — using (3) (and eventually further terms in $|y|^k$, for any k positive, according to the required precision in the Taylor approximation). We considered terms up to y^4 , since irrespective of the black hole horizon radius, the effective distance y along the extra dimension equals the compactification radius [15], as well as the effective size of the extra dimension probed by a graviton. Besides, our procedure considers suitable values for the post-Newtonian parameter in order that the Hawking radiation on the brane is zero. The number of degrees of freedom of KK gravitons is much less than the number of standard model particles in the Hawking radiation in the bulk, and the black hole energy irradiated into the KK modes must be a small fraction of the total luminosity. Since the post-Newtonian parameter β was chosen to prevent Hawking radiation in the brane, standard model particles on

the brane with high enough energy — larger than electroweak energy scale — are capable to overcome the confining mechanism [16]. In this case the bulk standard model fields should be included among the KK modes. Such mechanism responsible for the possible black hole recoil from the brane *corroborates* to the astrophysical phenomenology described in the figures above: the physical black hole radii $R_S = \sqrt{g_{\theta\theta}(R_S, 0)}$ are now *effectively* dislocated into the bulk, and given by $R_{\text{brane}} = \sqrt{g_{\theta\theta}(R_S, y)}$ in Eq.(13). Further discussion and details on the general behavior encoded in the figures are presented in the next Section.

V. CONCLUDING REMARKS AND OUTLOOK

Any phenomenologically successful theory in which our Universe is viewed as a brane must reproduce the large-scale predictions of general relativity on the brane. It implies that gravitational collapse of matter trapped on the brane provides the Casadio-Fabbri-Mazzacurati solutions on the brane: either a localized black hole or an extended black string solution, possessing a warped horizon. It is possible to intersect this solution with a vacuum domain wall and the induced metric is the ones presented in the analysis in Subsections III A and III B. In the case I, our analysis is restricted to the case where $\beta = 5/4$ since for this values there is a zero (Hawking) temperature black hole associated [13, 14]. Since we want to extract physical information on the braneworld effects on the variation of luminosity exclusively, we opted for this value for the parameter β , in such a way that the graphics, concerning this variation on the luminosity, take into account exclusively the braneworld effects, since Hawking radiation is shown to be suppressed with $\beta = 5/4$ for this metric. Analogously, for the case II the analysis is accomplished taking into account the value $\beta = 3/2$ in Eq.(10), as already discussed.

For the Casadio-Fabbri-Mazzacurati (types I and II) black holes the variation of luminosity of a supermassive black hole quasar due to the correction of the horizon in a braneworld scenario is given by $\Delta L \sim 10^{-3} L_\odot$. On the other hand, the Schwarzschild braneworld corrected black string horizon on the brane was previously investigated in [10], and in that case $\Delta L \sim 10^{-5} L_\odot$. It shows that the Casadio-Fabbri-Mazzacurati black hole solutions, containing the post-Newtonian parameter, can probe two orders of magnitude more the variation in the quasars luminosity.

Figs. 1 and 2 encode the brane effect-corrected black string horizon, respectively for the first and second black hole solution proposed by Casadio, Fabbri, and Mazzacurati, along the extra dimension y . The black string horizon behavior is obviously different for distinct values for the black hole mass. For the second case, the warped horizon is always an increasing function of the extra dimension. For the first one, instead, it holds only for a black hole with mass $M \gtrsim 73M_\odot$. Otherwise, the warped horizon is a decreasing function along the extra dimension.

Once the corrections related to the black string horizon behavior along the extra dimension are obtained, we focused on how such corrections can also alter the black string warped horizon. Our results are exposed and conflated in Figs. 3 and 4 illustrating the variation of luminosity of quasars — supermassive black holes — when the Hawking radiation in the brane is precluded, and the pure braneworld effect can be analyzed. The corrections of the luminosity regarding the Schwarzschild black string, along the extra dimension, can be probed by a black hole by recoil effects from the brane. The variation of the quasar luminosity is considerable in this case.

In [9] some properties of black holes were analyzed, in ADD [1] and Randall-Sundrum models. Mini black holes in ADD models have the first phase Hawking radiation mostly in the bulk and recoil effect to leave the brane. The analysis of the Casadio-Fabbri-Mazzacurati solutions in this paper sheds new light on mini black holes and their possible detection at LHC, since the preclusion of Hawking radiation can drastically modify the previous analysis about mini black holes in ADD and Randall-Sundrum braneworld models, as well as the mini black holes radiation in LHC measurements. The method here introduced can be immediately applied in such context.

Acknowledgments

R. da Rocha is grateful to Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) 476580/2010-2, 303027/2012-6, and 304862/2009-6 for financial support, and to Prof. J. M. Hoff da Silva for fruitful and valuable discussions and suggestions as well. A. M. Kuerten thanks to CAPES for financial support. C. H. Coimbra-Araújo thanks Fundação Araucária and Itaipu Binacional for financial support.

Appendix A: Kretschmann scalars for Casadio-Fabbri-Mazzacurati metrics

The 5D Kretschmann scalars associated to the black strings here discussed can be obtained by the 4D ones via Eq.(9).

Since we aim to investigate the particular case where $\beta = 5/4$ where there is no Hawking radiation, and pure braneworld effects can be probed, for this case the 4D Kretschmann scalar $K^{(I)} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ — where the Riemann tensors used are the ones related to the Casadio-Fabbri-Mazzacurati case I — is given by

$$\begin{aligned} K^{(I)} = & \left(\frac{GM}{c^2 r} \right)^4 \left[\frac{(1 - \frac{2GM}{c^2 r})}{(1 - \frac{3GM}{2c^2 r})} \right]^2 \left\{ \frac{GM}{c^2 r} \left(3 + \frac{7GM}{c^2 r} \right) \right\}^2 \\ & + 4 \left[\frac{8}{9} \left(1 - \frac{2GM}{c^2 r} \right)^2 + \frac{G^2 M^2}{2c^4 r^2 (1 - \frac{3GM}{2c^2 r})^2} \left[\left(1 - \frac{2GM}{c^2 r} \right) \left(\frac{5}{2} - \frac{3GM}{c^2 r} \right) \right]^2 + \left(1 - \frac{(1 - \frac{2GM}{c^2 r})^2}{(1 - \frac{3GM}{2c^2 r})} \right)^2 \right] \end{aligned} \quad (\text{A1})$$

which diverges at $r = \frac{3GM}{2c^2}$ and $r = 0$. It agrees with the result in [13, 14] where $K^{(I)} \propto (1 - \frac{3GM}{2c^2 r})^{-4}$ for values of r near to the respective singularity $r = \frac{3GM}{2c^2}$. Now, for the Casadio-Fabbri-Mazzacurati case II metric (10), the associated Kretschmann scalar, in the specific case here considered $\beta = 3/2$ the expression above is led to

$$\begin{aligned} K^{(II)} = & \left(\frac{GM}{c^2 r} \right)^2 \left[\frac{\left(1 - \frac{5GM}{2c^2 r} \right)}{\left(1 - \frac{3GM}{2c^2 r} \right) \left(1 - \frac{2GM}{c^2 r} \right)} \right]^2 \\ & \left[\left(\frac{3}{(1 - \frac{3GM}{2c^2 r})} - \frac{2}{(1 - \frac{2GM}{c^2 r})} - \frac{5}{2(1 - \frac{5GM}{2c^2 r})} \right) \frac{GM}{c^2 r} \left(1 - \frac{GM}{c^2 r} \right) - \frac{\frac{2GM}{c^2 r}}{(1 - \frac{2GM}{c^2 r})} - \left(\frac{6GM}{c^2 r} - 1 \right) \right]^2 \\ & + 4 \left[\frac{8}{9} \left(1 - \frac{2GM}{c^2 r} \right)^2 \left(1 - \frac{5GM}{2c^2 r} \right)^2 \left(1 - \frac{GM}{c^2 r} \right)^{-1} + \frac{\left(\frac{GM}{c^2 r} \right)^2}{2r^4 (1 - \frac{3GM}{2c^2 r})^2} \left[2 \left(1 - \frac{5GM}{2c^2 r} \right) \right. \right. \\ & \left. \left. + \frac{5}{2} \left(1 - \frac{2GM}{rc^2} \right) - \frac{3}{2} \left(1 - \frac{2GM}{rc^2} \right) \left(1 - \frac{5GM}{2rc^2} \right) \right]^2 + \frac{1}{r^4} \left(1 - \frac{\left(1 - \frac{2GM}{c^2 r} \right) \left(1 - \frac{5GM}{2c^2 r} \right)}{\left(1 - \frac{3GM}{2c^2 r} \right)} \right)^2 \right] \end{aligned} \quad (\text{A2})$$

[1] N. Arkani-Hamed, S. Dimopoulos, G. R. Dvali, *The Hierarchy Problem and New Dimensions at a Millimeter*, *Phys. Lett. B* **429** (1998) 263-272 [[arXiv:hep-ph/9803315](#)]; N. Arkani-Hamed, S. Dimopoulos, G. Dvali, *Phenomenology, Astrophysics and Cosmology of Theories with Sub-Millimeter Dimensions and TeV Scale Quantum Gravity*, *Phys. Rev. D* **59** (1999) 086004 [[arXiv:hep-ph/9807344](#)].

[2] V. A. Rubakov, M. E. Shaposhnikov, *Do we live inside a domain wall?*, *Phys. Lett. B* **125** (1983) 136; R. Gregory, V. A. Rubakov, S. M. Sibiryakov, *Brane worlds: the gravity of escaping matter*, *Class. Quant. Grav.* **17** (2000) 4437-4449 [[arXiv:hep-th/0003109](#)].

[3] P. Horava, E. Witten, *Heterotic and Type I String Dynamics from Eleven Dimensions*, *Nucl. Phys. B* **460** (1996) 506-524 [[arXiv:hep-th/9510209](#)]; P. Horava, E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, *Nucl. Phys. B* **475** (1996) 94 [[arXiv:hep-th/9603142](#)].

[4] K. R. Dienes, *String theory and the path to unification: a review of recent developments*, *Phys. Rep.* **287** (1997) 447-525 [[arXiv:hep-th/9602045](#)].

[5] L. Randall, R. Sundrum, *A large mass hierarchy from a small extra dimension*, *Phys. Rev. Lett.* **83** (1999) 3370-3373 [[arXiv:hep-ph/9905221](#)]; *An alternative to compactification*, *Phys. Rev. Lett.* **83** (1999) 4690-4693 [[arXiv:hep-th/9906064](#)]; M. Gogberashvili, *Our world as an expanding shell*, *Europhys. Lett.* **49** (2000) 396 [[arXiv:hep-ph/9812365](#)].

[6] A. Lukas, B. A. Ovrut, K. S. Stelle, D. Waldram, *The Universe as a Domain Wall*, *Phys. Rev. D* **59** (1999) 086001 [[arXiv:hep-th/9803235](#)]; *Cosmological Solutions of Horava-Witten Theory*, *Phys. Rev. D* **60** (1999) 086001 [[arXiv:hep-th/9806022](#)].

[7] J. M. Hoff da Silva and R. da Rocha, *Schwarzschild generalized black hole horizon and the embedding space*, *Eur. Phys. J. C* **72** (2012) 2258 [[arXiv:1212.2588 \[gr-qc\]](#)].

[8] J. M. Hoff da Silva, R. da Rocha, *Gravitational constraints of dS branes in AdS Einstein-Brans-Dicke bulk*, *Class. Quant. Grav.* **27** (2010) 225008 [[arXiv:1006.5176 \[gr-qc\]](#)]; M. C. B. Abdalla, J. M. Hoff da Silva, R. da Rocha, *Notes on the Two-brane Model with Variable Tension*, *Phys. Rev. D* **80** (2009) 046003 [[arXiv: 1101.4214 \[gr-qc\]](#)].

[9] R. da Rocha, C. H. Coimbra-Araujo, *Extra dimensions in CERN LHC via mini-black holes: effective Kerr-Newman braneworld effects*, *Phys. Rev. D* **74** (2006) 055006 [[arXiv:hep-ph/0607027](#)].

[10] R. da Rocha, C. H. Coimbra-Araujo, *Variation in the luminosity of Kerr quasars due to an extra dimension in the brane Randall-Sundrum model*, *JCAP* **0512** (2005) 009 [[arXiv:astro-ph/0510318](#)]; R. da Rocha and C. H. Coimbra-Araujo, *Could the variation in quasar luminosity, due to extra dimension 3-brane in rs model, be measurable?*, *Braz. J. Phys.*

35N4B (2005) 1129 [[arXiv:astro-ph/0509363](#)]; C. H. Coimbra-Araujo, R. da Rocha, I. T. Pedron, *Anti-de Sitter curvature radius constrained by quasars in brane-world scenarios*, *Int. J. Mod. Phys. D* **14** (2005) 1883 [[arXiv:astro-ph/0505132](#)].

[11] R. Gregory, R. Whisker, K. Beckwith, C. Done, *Observing braneworld black holes*, *JCAP* **1004** (2004) 013 [[arXiv:hep-th/0406252](#)]; G. L. Alberghi, R. Casadio, O. Micu, A. Orlandi, *Brane-world black holes and the scale of gravity*, *JHEP* **1109** (2011) 023 [[arXiv:1104.3043](#) [hep-th]].

[12] R. da Rocha, J. M. Hoff da Silva, *Black string corrections in variable tension braneworld scenarios*, *Phys. Rev. D* **85** (2012) 046009 [[arXiv:1202.1256](#) [gr-qc]].

[13] R. Casadio, A. Fabbri, L. Mazzacurati, *New black holes in the brane-world?*, *Phys. Rev. D* **65** (2002) 084040 [[arXiv:gr-qc/0111072](#)].

[14] R. Casadio, L. Mazzacurati, *Bulk shape of brane-world black holes*, *Mod. Phys. Lett. A* **18** (2003) 651 [[arXiv:gr-qc/0205129](#)].

[15] R. Maartens, K. Koyama, *Brane world gravity*, *Living Rev. Relativity* **13** (2010) 5 [[arXiv:1004.3962](#)].

[16] R. Casadio and B. Harms, *Black hole evaporation and compact extra dimensions*, *Phys. Rev. D* **64** (2001) 024016 [[arXiv:hep-th/0101154](#)].

[17] V. Cardoso, M. Cavaglia, L. Gualtieri, *Black hole particle emission in higher-dimensional spacetimes* *Phys. Rev. Lett.* **96** (2006) 071301, Erratum-ibid. **96** (2006) 219902 [[arXiv:hep-th/0512002](#)]; *Hawking emission of gravitons in higher dimensions: Non-rotating black holes* *JHEP* **0602** (2006) 021 [[arXiv:hep-th/0512116](#)].

[18] T. Shiromizu, K. Maeda, M. Sasaki, *The Einstein Equations on the 3-Brane World*, *Phys. Rev. D* **62** (2000) 043523 [[arXiv:gr-qc/9910076](#)]; A. N. Aliev and A. E. Gumrukcuoglu, *Gravitational Field Equations on and off a 3-Brane World*, *Class. Quant. Grav.* **21** (2004) 5081 [[arXiv:hep-th/0407095](#)].

[19] L. A. Gergely, R. Maartens, *Brane world generalizations of the Einstein static universe*, *Class. Quant. Grav.* **19** (2002) 213 [[arXiv:gr-qc/0105058](#)].

[20] W. Israel, *Singular hypersurfaces and thin shells in general relativity*, *Nuovo Cim.* **44 B** (1966) 1-14, Errata-ibid **48 B** [Series 10] (1967) 463; G. Darmois, *Memorial des sciences mathématiques*, Fascicule XXV ch V, Gauthier-Villars, Paris 1927.

[21] J. Lykken, L. Randall, *The shape of gravity*, *JHEP* **06** (2000) 014 [[arXiv:hep-th/9908076](#)].

[22] L. A. Gergely, *Black holes and dark energy from gravitational collapse on the brane*, *JCAP* **0702** (2007) 027 [[arXiv:hep-th/0603254](#)].

[23] N. Dadhich, R. Maartens, P. Papadopoulos, V. Rezania, *Black holes on the brane*, *Phys. Lett. B* **487** (2000) 1 [[arXiv:hep-th/0003061](#)].

[24] R. Casadio, C. Germani, *Gravitational collapse and black hole evolution: Do holographic black holes eventually anti-evaporate?* *Prog. Theor. Phys.* **114** (2005) 23 [[arXiv:hep-th/0407191](#)].

[25] D. Jennings, I. R. Vernon, A. C. Davis, C. van de Bruck, *Bulk black holes radiating in non-Z(2) brane-world spacetimes*, *JCAP* **0504** (2005) 013. [[arXiv:hep-th/0412281](#)].

[26] S. S. Seahra, C. Clarkson and R. Maartens, *Detecting extra dimensions with gravity wave spectroscopy: the black string brane-world*, *Phys. Rev. Lett.* **94** (2005) 121302 [[arXiv:gr-qc/0408032](#)].

[27] M. Bruni, C. Germani, R. Maartens, *Gravitational collapse on the brane: a no-go theorem*, *Phys. Rev. Lett.* **87** (2001) 231302 [[arXiv:gr-qc/0108013](#)]; T. Tanaka, *Classical black hole evaporation in Randall-Sundrum infinite braneworld*, *Prog. Theor. Phys. Suppl.* **148** (2002) 307 [[arXiv:gr-qc/0203082](#)]; R. Emparan, A. Fabbri, N. Kaloper, *Quantum Black Holes as Holograms in AdS Braneworlds*, *JHEP* **08** (2002) 043 [[arXiv:hep-th/0206155](#)].

[28] C. Germani, R. Maartens, *Stars in the braneworld*, *Phys. Rev. D* **64** (2001) 124010 [[arXiv:hep-th/0107011](#)].

[29] T. Shiromizu, M. Shibata, *Black holes in the brane world: Time symmetric initial data*, *Phys. Rev. D* **62** (2000) 127502 [[arXiv:hep-th/0007203](#)].

[30] R. Gregory, *Black string instabilities in anti-de Sitter space*, *Class. Quant. Grav.* **17** (2000) L125 [[arXiv:hep-th/0004101](#)].

[31] I. Chamblin, S. Hawking, H. S. Reall, *Brane-World Black Holes*, *Phys. Rev. D* **61** (2000) 065007 [[arXiv:hep-th/9909205](#)].

[32] P. Kanti, *Black Holes in Theories with Large Extra Dimensions: a Review*, *Int. J. Mod. Phys. A* **19** (2004) 4899-4951 [[arXiv:hep-ph/0402168](#)].

[33] N. Deruelle, *Stars on branes: the view from the brane* [[arXiv:gr-qc/0111065](#)].

[34] M. Visser, D. L. Wiltshire, *On-brane data for braneworld stars*, *Phys. Rev. D* **67** (2003) 104004 [[arXiv:hep-th/0212333](#)].

[35] I. Giannakis, H. Ren, *Possible extensions of the 4D Schwarzschild horizon in the 5D brane world*, *Phys. Rev. D* **63** (2001) 125017 [[arXiv:hep-th/0010183](#)].

[36] P. Kanti, K. Tamvakis, *Quest for Localized 4-D Black Holes in Brane Worlds*, *Phys. Rev. D* **65** (2002) 084010 [[arXiv:hep-th/0110298v2](#)]; P. Kanti, I. Olasagasti, K. Tamvakis, *Quest for Localized 4-D Black Holes in Brane Worlds. II : Removing the bulk singularities*, *Phys. Rev. D* **68** (2003) 124001 [[arXiv:hep-th/0307201](#)].

[37] J. Garriga, T. Tanaka, *Gravity in the Randall-Sundrum Brane World*, *Phys. Rev. Lett.* **84** (2000) 2778 [[arXiv:hep-th/9911055](#)]; H. Kudoh and T. Tanaka, *Second order perturbations in the Randall-Sundrum infinite brane world model*, *Phys. Rev. D* **64** (2001) 084022 [[arXiv:hep-th/0104049](#)].

[38] D. J. Kapner, *Tests of the Gravitational Inverse-Square Law below the Dark-Energy Length Scale*, *Phys. Rev. Lett.* **98** (2007) 021101; R. Casadio, O. Micu, *Exploring the bulk of tidal charged micro-black holes*, *Phys. Rev. D* **81** (2010) 104024 [[arXiv:1002.1219](#) [hep-th]].

[39] J. Ahmed, K. Saifullah, *Hawking radiation of Dirac particles from black strings*, *JCAP* **1108** (2011) 011 [[arXiv:1108.2677](#) [hep-th]].

[40] R. Emparan, J. Garcia-Bellido, N. Kaloper, *Black hole astrophysics in AdS braneworlds*, *JHEP* **0301** (2003) 079 [[arXiv:hep-th/0212132](#)].

- [41] C. D. Hoyle, U. Schmidt, B. R. Heckel, E. G. Adelberger , J. H. Gundlach, D. J. Kapner, H. E. Swanson, *Submillimeter Test of the Gravitational Inverse-Square Law: A Search for "Large" Extra Dimensions*, *Phys. Rev. Lett.* **86** (2001) 1418 ; J. C. Long, H. W. Chan, A. B. Churnside, E. A. Gulbis, M. C. M. Varney, J. C. Price, *Upper limits to submillimetre-range forces from extra space-time dimensions*, *Nature* **421** (2003) 922.
- [42] S. B. Giddings, E. Katz, L. Randall, *Linearized Gravity in Brane Backgrounds*, *JHEP* **0003** (2000) 023 [[arXiv:hep-th/0002091](https://arxiv.org/abs/hep-th/0002091)].
- [43] X. H. Ge, S. W. Kim, *Black hole analogues in braneworld scenario* [[arXiv:0705.1396 \[hep-th\]](https://arxiv.org/abs/0705.1396)].
- [44] S. L. Shapiro, S. A. Teukolski, *Black holes, white dwarfs, and neutron stars*, Wiley-Interscience, New York 1983.
- [45] V. P. Frolov and D. Stojkovic, *Black hole radiation in the brane world and recoil effect* *Phys. Rev. D* **66** (2002) 084002 [[arXiv:hep-th/0206046](https://arxiv.org/abs/hep-th/0206046)]; V. P. Frolov and D. Stojkovic, *Black hole as a point radiator and recoil effect in the brane world*, *Phys. Rev. Lett.* **89** (2002) 151302 [[arXiv:hep-th/0208102](https://arxiv.org/abs/hep-th/0208102)].
- [46] D. Stojkovic, *Distinguishing between the small ADD and RS black holes in accelerators*, *Phys. Rev. Lett.* **94** (2005) 011603 [[arXiv:hep-ph/0409124](https://arxiv.org/abs/hep-ph/0409124)].
- [47] D. -C. Dai, C. Issever, E. Rizvi, G. Starkman, D. Stojkovic and J. Tseng, *Manual of BlackMax, a black-hole event generator with rotation, recoil, split branes, and brane tension*, [[arXiv:0902.3577 \[hep-ph\]](https://arxiv.org/abs/0902.3577)]; D. -C. Dai, G. Starkman, D. Stojkovic, C. Issever, E. Rizvi and J. Tseng, *BlackMax: A black-hole event generator with rotation, recoil, split branes, and brane tension*, *Phys. Rev. D* **77** (2008) 076007 [[arXiv:0711.3012 \[hep-ph\]](https://arxiv.org/abs/0711.3012)].
- [48] D. N. Page, *Particle Emission Rates from a Black Hole: Massless Particles from an Uncharged, Nonrotating Hole*, *Phys. Rev. D* **13** (1976) 198; D. N. Page, *Particle Emission Rates from a Black Hole. 2. Massless Particles from a Rotating Hole*, *Phys. Rev. D* **14** (1976) 3260.